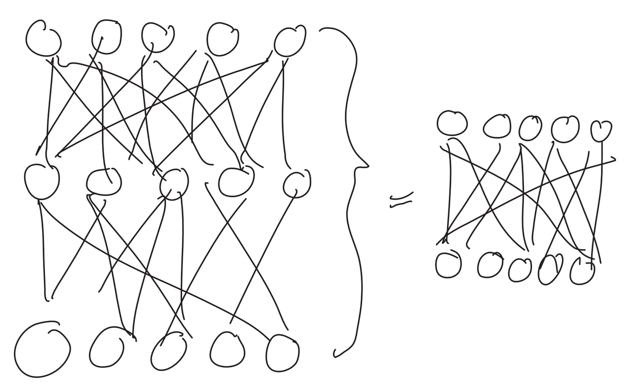
Linear Networks

Feedforward

$r_{out} = W r_{in}$

Cascade of feedforward linear networks is equivalent to a single feedforward linear net.



$r_1 \to r_2 \to r_3$

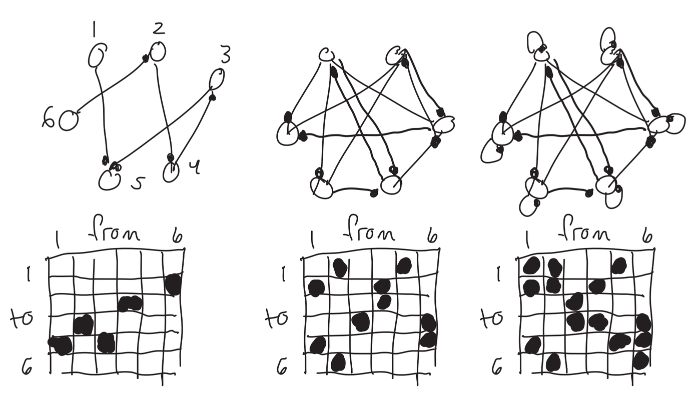
$r_2 = A r_1 \quad r_3 = B r_2$

$r_3 = BA r_1$

Recurrent Linear Network

$\dot{r} = -r + Wr$

$\dot{r}_i = -r_i + \sum_j W_{ij} r_j$

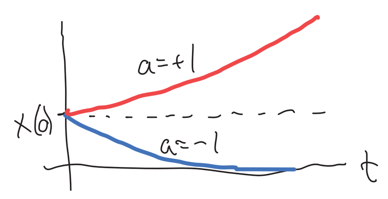


Refresher: 1st order differential equations

Scalar, homogeneous case

$\dot{x} = ax$

$x(t) = e^{at} x(0)$



inhomogeneous case

$\dot{x} = ax + b(t)$

$x(t) = e^{at} x(0) + \int_0^t e^{a(t-t')} b(t')\,dt'$

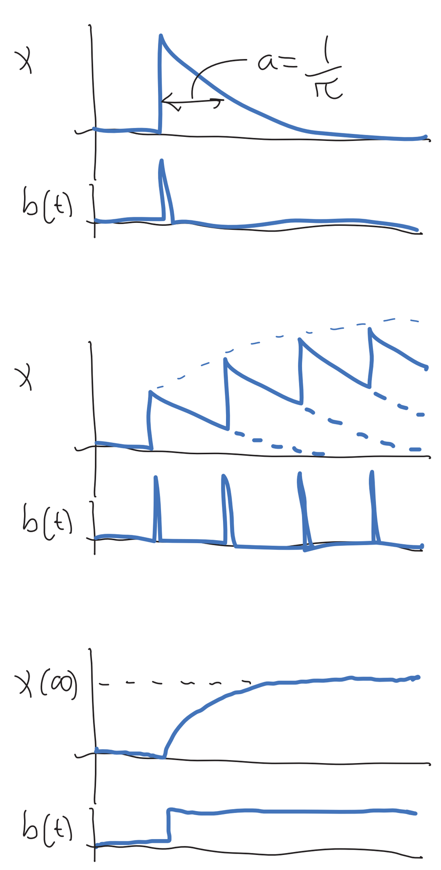
$x(t) = e^{at} x(0) + e^{at} \int_0^t e^{-at'} b(t')\,dt'$

$\dot{x} = a e^{at} x(0) + a e^{at} \int_0^t e^{-at'} b(t')\,dt' + \cancel{e^{at}}\,\cancel{e^{-at}}\, b(t)$

$= a\left[e^{at} x(0) + \int_0^t e^{a(t-t')} b(t')\,dt'\right] + b(t)$

$= a x(t) + b(t)$ ✓

$x(t) = e^{at} x(0) + \underbrace{\left[e^{-at} \circ b\right]}_{\text{exponential moving average (convolution)}}(t)$



Homogeneous case: no input

$\dot{r}_i = -r_i + \sum_j W_{ij} r_j$

$\dot{r} = \underbrace{(W - I)}_{A = Q \Lambda Q^{-1}} r$

def: $\rho = Q^{-1} r$

$Q^{-1}\dot{r} = Q^{-1} A r = Q^{-1} Q \Lambda Q^{-1} r$

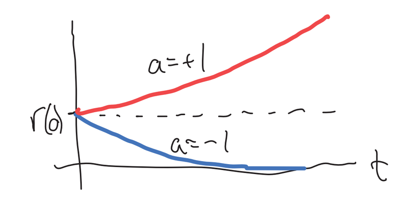
$\dot{\rho} = \Lambda \rho \to \rho_i(t) = e^{\lambda_i t} \rho_i(0)$

$r(t) = Q e^{\lambda_i t} Q^{-1} r(0)$

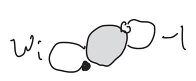
$= e^{At} r(0)$

Diagonal A
($a_i = A_{ii}$)

$r_i(t) = e^{a_i t} r_i(0)$



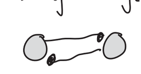
$a_i > 0 \Rightarrow w_i > 1$



w_i -1 (decay)

Timescale $= \frac{1}{|a|}$

Symmetric A
$A_{ij} = A_{ji}$



All eigenvalues are real, $\mathrm{Im}(\lambda_k) = 0$

Eigenvectors are orthogonal

For each mode there is growth or decay

Antisymmetric A
$A_{ij} = -A_{ji}$

$\Downarrow$

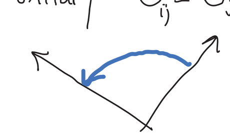
$A_{ii} = 0$

(no decay for each neuron)

All eigenvalues are imaginary, $\mathrm{Re}(\lambda_k) = 0$

Eigenvectors are unitary $U_{ij} = \bar{U}_{ji}$

Pure rotation:



e^{At} is an orthogonal matrix, $|e^{At}|_m = 1$

For each mode (complex) we have $\rho_k(t) = \rho_k(0) e^{i\omega_k t}$

$\omega_k = \mathrm{Im}(\lambda_k)$

Can also write as pairs of real modes

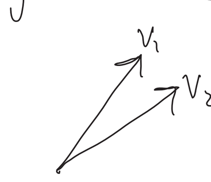
$\Lambda = \begin{pmatrix} i\omega_k & & \\ & \ddots & \\ & & \ddots \end{pmatrix}$ or $\begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix} = \Omega$

$A = U \Lambda U^{-1}$ $\qquad$ $A = O \Omega O^{-1}$

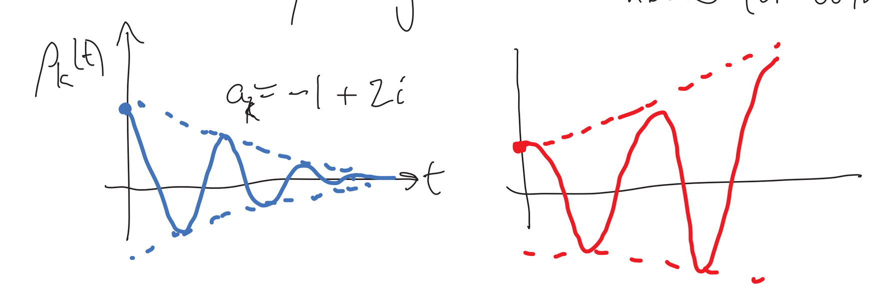
General A

General A ← Normal matrices: diagonalized by U

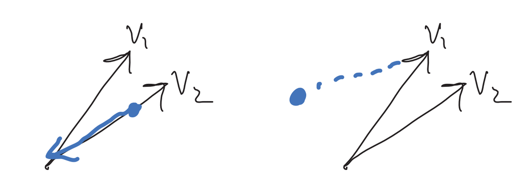
General A ← Nonnormal matrices: eigenvectors not orthogonal



Normal — most intuition, modes act independently, decay or grow or oscillate (or both)



Nonnormal — transient behaviors possible



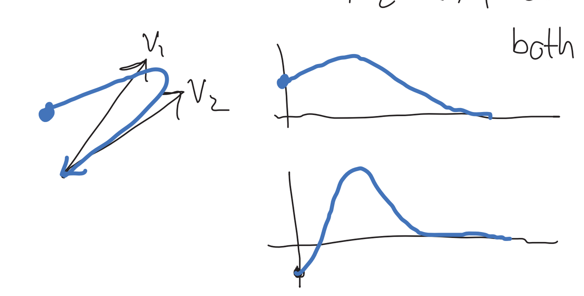
$r(0) = c v_2$

$r(t) = c e^{\lambda_2 t} v_2$

$r(0) = v_1 - v_2$

$r(t) = e^{\lambda_1 t} v_1 - e^{\lambda_2} v_2$

If $\lambda_2 \ll \lambda_1$ but even if both are real:

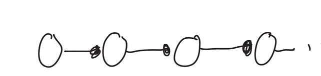


Equivalent to a feedforward structure on activity modes: can always express matrix in Jordan Normal Form:

$$\begin{pmatrix} \begin{pmatrix} \lambda_1 & a_{12} & \\ & \lambda_2 & a_{23} \\ & & \lambda_3 \end{pmatrix} & & & \\ & (\lambda_4) & & \\ & & \begin{pmatrix} \lambda_5 & a_{56} \\ & \lambda_6 \end{pmatrix} & \\ & & & \ddots \end{pmatrix}$$

Feedforward chain (bucket brigade)

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$



Dale's law: Neurons only make excitatory OR inhibitory synapses, NOT BOTH

(This allows us to categorize neurons as excitatory or inhibitory)

$W = \begin{pmatrix} W_{EE} & -W_{EI} \\ W_{IE} & -W_{II} \end{pmatrix}$ where $W_{ij} > 0 \;\forall i,j$ ← Nonnormal

More later on this.

What about input?

Inhomogeneous case:

$\dot{r} = -r + Ar + b(t)$

$r(t) = \underbrace{\int e^{A(t-t')} b(t')\,dt'}_{\text{filtered input}} + e^{At} r(0)$

$= [f \circ b](t) + e^{At} r(0)$

convolution with $f = e^{At}$