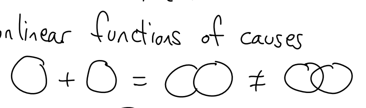
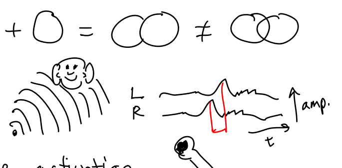
Nonlinear Networks + Dynamical Systems

All interesting computation is nonlinear:

- Sensory inputs are nonlinear functions of causes
  - Occlusion in vision



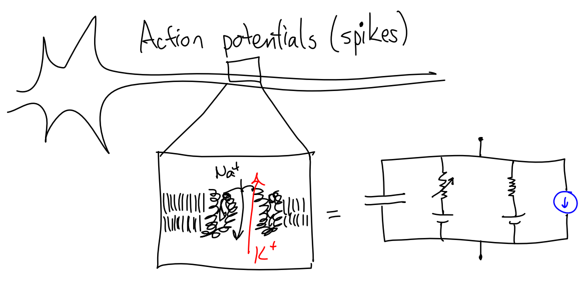
  - Sound localization in audition



- Actions are nonlinear in muscle actuation

Neural responses are nonlinear

Action potentials (spikes)



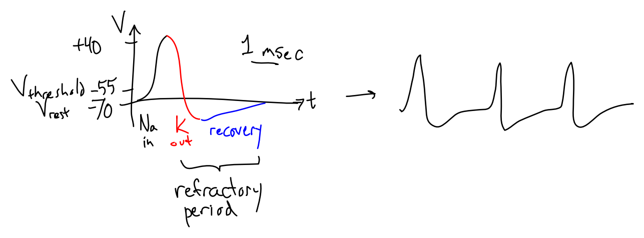
Hodgkin-Huxley model for action potential:

$$I = C_m\dot V + \bar g_K n^4 (V-V_K) + \bar g_{Na} m^3 h (V-V_{Na}) + \bar g_L (V-V_L)$$

$\dot n = \alpha_n(1-n)+\beta_n n$ K

$\dot m = \alpha_m(1-m)+\beta_m m$ Na activation

$\dot h = \alpha_h(1-h)+\beta_h h$ Na inactivation



Simple model: leaky integrate and fire

$I = C\dot V + g_L(V-V_L)$ no active conductance, g_L is indep of V, t

$V(t)=\Theta \longrightarrow V(t^+)=V_{reset}$ and generate spike at time t

change variables:

$g_L(V-V_L)=U$ so $\dot U = g_L\dot V$

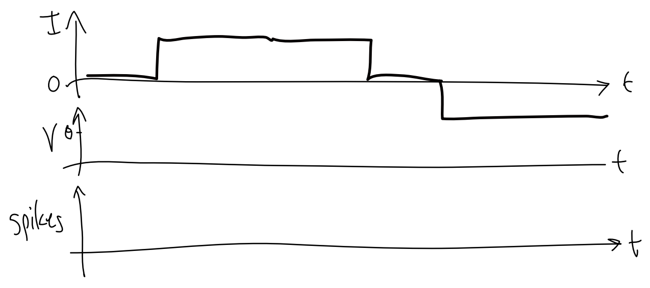
$I = \frac{C}{g_L} g_L\dot V + U = \tau\dot U + U$

$$\boxed{\begin{array}{l}\tau\dot V = -V + I \qquad \text{Leaky I\&F}\\ V(t)=1 \longrightarrow V(t^+)=V_r\end{array}}$$

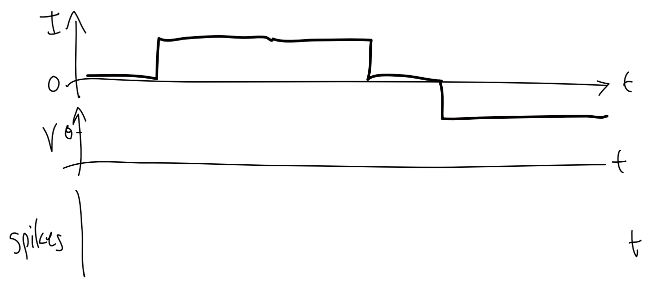
Simpler model: no leak, $\tau\to\infty$, rescale I, $V_r=0$

$\dot V = I$

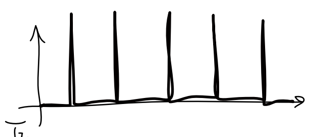
$V_t=1 \to V_{t^+}=0$ and generate a spike at t



Leaky integrate and fire



$$\rho(t) = \sum_{k=1}^{N_{spikes}} \delta(t-t_k)$$

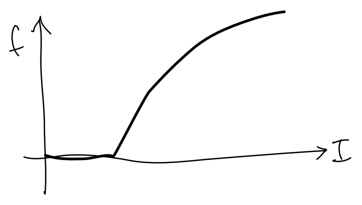


spike count $N(T_1,T_2) = \int_{T_1}^{T_2} dt\, r(t)$

firing rate $r(t) = \frac{1}{\Delta T}\int_{T_1}^{T_1+\Delta T} dt\, r(t)$ (in Hertz = 1/sec)

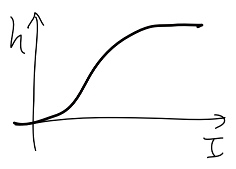
"Timing codes vs. rate codes" debate hinges on what is correct Δt?

f-I curve $f=r(t)$ firing rate



Rate neuron:

$\tau\dot r = -r + h(I)$



LNP neuron: spiking model based on rate neuron

$\rho \sim \text{Poiss}(r(t)\Delta t)$ (more on this later)

Rate Networks

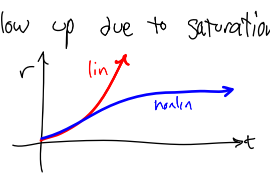
$$\tau\dot r_i = -r_i + h\Big(\sum_j W_{ij} r_j(t) + I_i(t)\Big)$$

vector notation: $\tau\dot r = -r + h(Wr + I)$

Almost the same as linear networks except for h, which makes a crucial difference!

In stable W regime, there is little change. But for eigenvalues λ_k of $A = W - I$, if $\lambda_k > 0$ then the system does not blow up due to saturation!

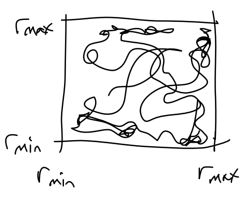
One neuron



$\dot r = -r + 2r$

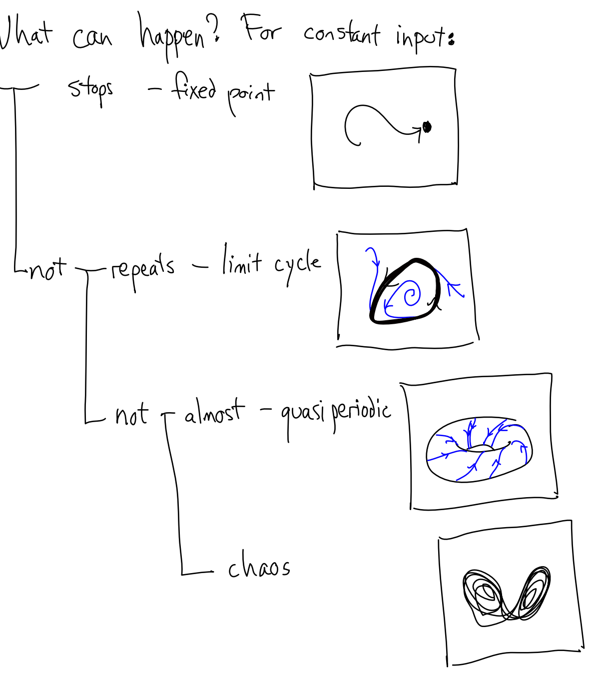
$\dot r = -r + \tanh 2r$

All firing rates are confined to a box:



What can happen? For constant input:

- stops – fixed point
- not
  - repeats – limit cycle
  - not
    - almost – quasi periodic
    - chaos



Fixed points: $\dot r = 0$

$\dot r = 0 = -r + h(Wr + I)$

$r = h(Wr + I)$ self-consistent eqn solved at r^*

Consider local expansion, $r = r^* + \delta r$, and look at locally linear approx:

$\delta\dot r = -r^* - \delta r + h(W(r^*+\delta r) + I)$

$\approx \cancel{-r^*} - \delta r + \cancel{h(Wr^*+I)} + H' W\delta r \quad (H'_{ij} = \delta_{ij} h'_i)$

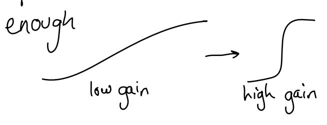
$\delta\dot r = -\delta r + H'W\delta r = B\delta r \qquad B = H'W - I$

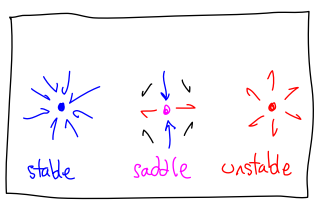
Linear dynamics: evaluate stability by eigenvalues β of B

Note that H' can change stability!

1D case: $\beta = h'w - 1$ so iff $h' > \frac{1}{w}$ then $\beta > 0$!

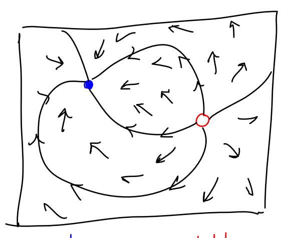
Thus the nonlinearity may destabilize a network's fixed pt. if gain is high enough



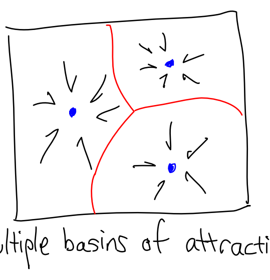


Multiple fixed points

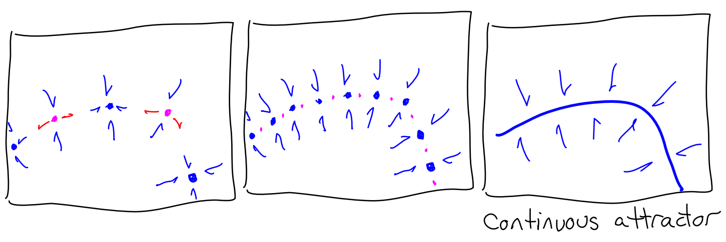
(useful to plot $\dot r$ as vector field. Trajectories $r(t)$ follow flow.)



stable and unstable fixed points



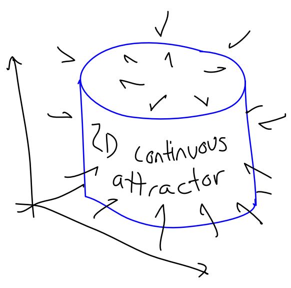
multiple basins of attraction



continuous attractor

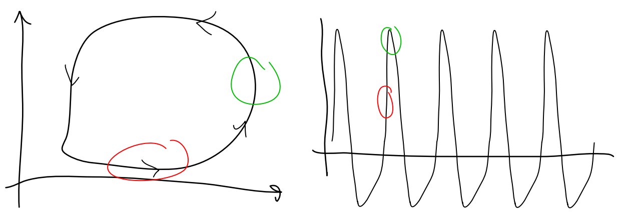
$\beta_\parallel = 0$, $\beta_\perp < 0$

(requires fine tuning)



Limit Cycles

For some inputs, H-H neuron has limit cycles



Variation: heteroclinic orbit

Connects saddle points

Changing input

The emergence of a new behavior with a parameter (like input current) is described by <u>bifurcation theory</u>

I = small

Hopf bifurcation

large (action potential)